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XXIII. *On the Equilibrium of a Mass of Homogeneous Fluid at liberty.* By JAMES IVORY, K.H. M.A. F.R.S., *Instit. Reg. Sc. Paris. Corresp., et Reg. Sc. Götting. Corresp.*

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OF the questions to which the publication of the *Principia* gave rise, none has been attended with greater difficulty than that which relates to the figure of the planets. In this research it is required to determine the figure of equilibrium of a mass of fluid consisting of particles that mutually attract one another at the same time that they are urged by a centrifugal force caused by a rotation about an axis. Geometers have long ago adopted a theory of the equilibrium of fluids which is said to be perfect, and to leave only mathematical difficulties to be surmounted in every problem: but it must be admitted that the utility of this theory amounts to very little; for it has failed in solving the fundamental problem for determining the figure of equilibrium of a homogeneous planet in a fluid state. This is the more remarkable, because MACLAURIN, soon after the origin of such inquiries, demonstrated with accuracy and elegance, that a planet supposed fluid would be in equilibrium if it had the figure of an oblate elliptical spheroid. To every one that reflects, the question, not easily answered, must occur, Why has it been found impossible to deduce the discovery of MACLAURIN from the analytical theory? If we suppose that the theory is physically correct, and that mathematical difficulties alone oppose its successful application, there is great probability that these would have yielded, as in other instances, to the repeated attempts of geometers.

But if CLAIRAUT's theory of the equilibrium of fluids be examined attentively and without prejudice, other difficulties of greater moment will present themselves. In a homogeneous fluid at liberty, if the forces in action be such as to make the problem possible, the equilibrium, according to the theory, requires only one condition, namely, that the forces urging every particle in the surface be directed perpendicularly towards that surface. The solution is thus made to depend entirely upon the differential equation of the surface, and seems to demand that this equation be determinate, and explicitly given: for if the equation be indeterminate, or not explicitly given, how can it be said that the problem is solved? If the forces which urge the particles of the fluid are explicit functions of the coordinates of the point on which they act, so that when the values of the coordinates are assigned, the algebraic expressions are completely ascertained, there is no doubt that the equation of the fluid's surface will be known, and the figure of equilibrium will be determined. With respect to such problems, the

theory of CLAIRAUT is therefore perfect, and it possesses all the elegance which might be expected from the talents of the author. On the other hand, if the forces in action depend upon the very figure to be found, as must always happen when the particles attract one another, the equation of the surface will not be explicitly known, because the differential coefficients are derived, in part at least, from the unknown figure of the fluid. Since quantities which depend entirely upon what is sought are not eliminated from the final equation, the ordinary rules of mathematical investigation would lead us to infer, either that the problem is not solved, or that it is indeterminate, and admits of many solutions. It is allowed on all hands that there is a mutual connexion between the figure of a mass of fluid and the attractions it exerts upon its particles: the relation which these two things, alike unknown, must bear to one another in the case of equilibrium, is expressed by the equations of the upper surface and of the interior level surfaces; and therefore it seems hardly possible to deny that these equations are indeterminate. What is wanting to complete the solution of the problem cannot possibly be supplied by any abstract or mathematical properties which the indeterminate equations may possess; and hence arises a suspicion that there is an imperfection of the theory, proceeding, probably, from some necessary condition having been overlooked.

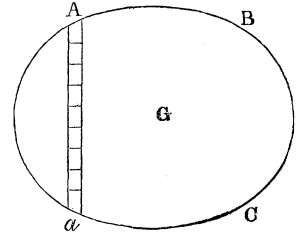
None of the observations that have been made go the length of charging with inaccuracy any of the properties of CLAIRAUT's theory, or any of the equations which express those properties. An equilibrium of a mass of fluid entirely at liberty cannot exist, unless all the conditions of that theory be fulfilled. The question is, whether those conditions be sufficient to determine completely the figure of equilibrium in all hypotheses respecting the forces. It is no small imperfection that the principal points of this theory have not been deduced from the nature of an equilibrium in a manner independent of opinion or arbitrary assumptions. If a strict mode of investigation had been followed, we should have been in possession of a just criterion for ascertaining in any particular case, whether all the conditions required for an equilibrium were fulfilled or not. But in solving problems of this kind, it is often thought sufficient to prove some enumerated properties, or to obtain certain algebraic equations, which unavoidably introduces obscurity and occasions a want of evidence; since it can hardly be supposed that the same properties, or the same equations, will bear alike upon a great variety of problems differing from one another in the nature of the forces urging the particles of a fluid.

Is not the principle, that the equilibrium of a mass of fluid is in all cases secured when every individual particle is pressed equally by all the canals issuing from it and terminating in the surface, an opinion or an assumption? That the property is general, no one will doubt. But when the fluid consists of attracting particles, the forces urging the particles and the pressures of the canals both vary when the upper surface of the fluid is made to change: and may it not be alleged that the variation of the figure of the mass may be such that the pressures of all the canals may still

continue to be equal? Thus it may be possible that the assumed principle may be fulfilled in the same body of fluid under different forms.

The difficulties which must be overcome before this subject can be freed from inaccurate and insufficient reasoning, have occurred in problems relating to fluids of uniform density; and for this reason homogeneous fluids are alone treated of in what follows.

1. Suppose that  $ABC$  represents a mass of homogeneous fluid entirely at liberty, the particles of which are urged by accelerating forces; let all the forces which act upon any element of the mass, as  $dm$ , be reduced to the directions of three rectangular coordinates  $x, y, z$ ; and put  $X, Y, Z$  for the sums of the partial forces respectively parallel to  $x, y, z$ . Now, if  $Aa$  be an infinitely slender prism of the fluid parallel to  $x$ , passing completely through the mass, and divided in its whole length into elementary portions, it is obviously a condition necessary to the equilibrium of the body of fluid, that the forces  $X$ , acting upon all the elements of  $Aa$ , mutually destroy one another.



What has been enunciated of a prism parallel to  $x$ , must hold equally of prisms parallel to  $y$  and  $z$ .

Any element  $dm$  may be conceived as formed by the intersection of three slender prisms parallel to  $x, y, z$ ; and, as the pressures in the whole extent of each prism balance another, the element will be at rest, having no tendency to move parallel to  $x$ , or to  $y$ , or to  $z$ . But no proof is required to show that an elementary portion of a fluid in equilibrium must be pressed equally on all sides.

The forces which act upon the elements at the ends of any prism,  $Aa$ , passing completely through the mass parallel to  $x$ , are necessarily directed inward, and have opposite directions; wherefore the force  $X$ , in varying through the whole length of  $Aa$ , must first decrease, then become equal to zero, and afterwards changing its sign, increase in approaching the other surface of the fluid. Thus, in every slender prism parallel to  $x$ , there is a point at which the force  $X$  is equal to zero; and if the whole body of fluid be divided into such prisms, all the zero points will form a continuous surface stretching completely through the mass. In like manner there will be two other internal surfaces containing all the points at which the forces  $Y$  and  $Z$  are evanescent. The intersection of the three surfaces will determine a point  $G$  within the body of fluid at which all the three forces  $X, Y, Z$ , vanish, and which may be called the centre of the mass in equilibrium.

In considering the equilibrium of a mass of fluid entirely at liberty, it is obvious that we may abstract from any motion common to all the particles, and from any forces acting upon them all with equal intensity in the same direction. The forces that must be balanced and rendered ineffective to produce motion, are such only as tend to change the relative position of the particles with respect to one another;

which supposes that the centre of gravity of the whole body of fluid continues at rest and free from the action of any forces. Thus it appears that G, the only point of a fluid in equilibrium not acted upon by any force, is no other than the centre of gravity of the mass.

2. The equilibrium of a fluid entirely at liberty will not be disturbed by a pressure of the same intensity applied to all the parts of the exterior surface.

By the intensity of a pressure is meant the amount of it when applied to some given surface, most conveniently to the unit of surfaces. A constant pressure, or one acting uniformly with the same intensity, is proportional to the surface to which it is applied.

This being understood, what is affirmed above is an immediate consequence of the fundamental property of an incompressible fluid to transmit a pressure exerted upon its surface in all directions without any loss of intensity. The inward pressure upon any part of the surface thus produces an equivalent outward pressure upon every other part, which is balanced by the contrary pressure supposed to act over the whole surface. Wherefore if a mass of fluid be in equilibrium, it will continue in equilibrium, supposing a pressure of the same intensity to be applied to all parts of the surface.

3. The action of the forces upon the particles in the interior parts of the body of fluid is next to be considered.

Take any point  $(x\ y\ z)$  of the mass, and draw through it in any direction a plane surface  $w$  infinitely small and of any figure; from the same point  $(x\ y\ z)$  draw the infinitely short line  $\delta s$  perpendicular to  $w$ , and construct an upright prism upon the base  $w$  with the height  $\delta s$ . The forces acting upon a particle at the point  $(x\ y\ z)$  being represented as before by  $X, Y, Z$ , and the coordinates of the end of  $\delta s$  being  $x + \delta x, y + \delta y, z + \delta z$ , we shall have this identical equation,

$$\left(X \frac{\delta x}{\delta s} + Y \frac{\delta y}{\delta s} + Z \frac{\delta z}{\delta s}\right) \times \delta s \times w = (X \delta x + Y \delta y + Z \delta z) \times w;$$

or by introducing a new symbol,

$$F = X \frac{\delta x}{\delta s} + Y \frac{\delta y}{\delta s} + Z \frac{\delta z}{\delta s},$$

$$F \times \delta s \times w = (X \delta x + Y \delta y + Z \delta z) \times w.$$

Now  $\frac{\delta x}{\delta s}, \frac{\delta y}{\delta s}, \frac{\delta z}{\delta s}$ , are the cosines of the angles which the directions of the forces make with  $\delta s$ : wherefore  $X \frac{\delta x}{\delta s}, Y \frac{\delta y}{\delta s}, Z \frac{\delta z}{\delta s}$ , are the partial forces urging the particle  $(x\ y\ z)$  in the direction of  $\delta s$ ; and the whole accelerating force in the same direction is equal to  $F$ . The density being constant, and represented by unit, the mass of the prism will be equal to  $\delta s \times w$ ; and as this may be as small as we please, we may assume that every particle of it is urged by the same force  $F$ ; so that  $F \times \delta s \times w$  is the effort of the prism to move from the point  $(x\ y\ z)$  in the direction of  $\delta s$ . Let  $p$ , a

function of  $x, y, z$ , represent the intensity of pressure at the point  $(x y z)$ , and  $p + \delta p$  will be the intensity at the other end of  $\delta s$ : the external pressures acting upon the opposite ends of the prism are therefore  $p \times w$  and  $(p + \delta p) \times w$ ; and the difference of these, or  $\delta p \times w$ , is the impulse causing the prism to move towards the point  $(x y z)$  in the direction of  $\delta s$ . Now, the prism being at rest, the impulses  $F \times \delta s \times w$  and  $\delta p \times w$ , which tend to move it in opposite directions, must be equal; wherefore, taking the foregoing value of  $F \times \delta s \times w$ , and suppressing the factor  $w$ , which is common to the equal quantities, the non-effect of the opposite forces requires this equation,

$$-\delta p = X \delta x + Y \delta y + Z \delta z,$$

which expresses that the effort of the accelerating forces to move the prism in any direction is counterbalanced by the contrary action of the pressure. The equation must hold at every point of the mass, without any relation being supposed between the infinitely small quantities  $\delta x, \delta y, \delta z$ ; which condition requires that

$$X \delta x + Y \delta y + Z \delta z$$

be the variation of a function in which the three variables  $x, y, z$ , are independent of one another. If this function be represented by  $\phi'(x, y, z)$ , so that

$$\int (X dx + Y dy + Z dz) = \phi'(x, y, z),$$

we shall have

$$C - p = \phi'(x, y, z).$$

The forces respectively parallel to  $x, y, z$ , are now thus expressed:

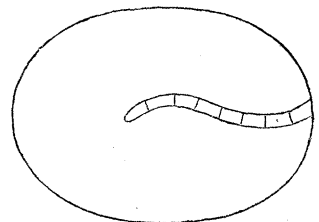
$$X = \frac{d \cdot \phi'(x, y, z)}{dx}, \quad Y = \frac{d \cdot \phi'(x, y, z)}{dy}, \quad Z = \frac{d \cdot \phi'(x, y, z)}{dz}.$$

The differentials of  $\phi'(x, y, z)$  vanishing at the centre of gravity, the function will increase on every side in receding from that point; and when it becomes equal to  $C$ , we shall have

$$C = \phi'(x, y, z),$$

which is the equation of the surface of the fluid, the pressure  $p$  being equal to zero at all the points of that surface.

If an infinitely narrow canal of any figure be extended from the point  $(x y z)$  to the surface of the fluid, the intensity with which all the fluid in the canal presses at the point  $(x y z)$  will be equal to the function  $p$ . Let the whole length of the canal be divided into small parts,  $\delta s, \delta s', \delta s''$ , &c.; and at every point of division draw the sections  $w, w', w''$ , &c., perpendicular to the sides of the canal, which will thus be divided into an infinite number of small prisms, to every one of which the foregoing investigation will apply. Wherefore, the variation of the intensity of pressure, or  $\delta p$ , in the length of any prism, will be just equal to the action of the accelerating forces upon the particles of the prism; and the intensity with which all the fluid in the



canal presses at the point  $(x\ y\ z)$  will be equal to the sum of all the variations of the function  $p$  in the whole length of the canal, that is, to the difference between the value of  $p$  at the point  $(x\ y\ z)$  and at the surface of the fluid. Now the value of  $p$  at the surface of the fluid is equal to zero; wherefore, the intensity with which all the fluid in the canal presses at the point  $(x\ y\ z)$  is equal to the value of  $p$  at that point.

It follows from what has been proved, that every narrow canal drawn from any point  $(x\ y\ z)$ , and terminating in the surface of the fluid, will press at that point with equal intensity. Hence, if an infinitely small mass of the fluid, such as a sphere, or a cube, &c., be situated at the point  $(x\ y\ z)$ , it will have no tendency to move by the action of the surrounding fluid; for it will be equally pressed by all the narrow canals standing upon different portions of its surface, and extending to the surface of the fluid. This property is perfectly general and necessary; and it may become a question, whether it be not alone sufficient to secure an equilibrium. Without entering upon the discussion of this question, we here confine our attention strictly to what has been demonstrated, namely, that in a fluid in equilibrium, every infinitely small portion is pressed with equal intensity by all the narrow canals issuing from it, and terminating in the surface of the fluid\*.

4. According to what has been shown, the forces which urge the particles of a fluid in equilibrium, and the consequent pressures, depend upon one function  $\phi'(x, y, z)$ , varying in its value as the coordinates change their place from the centre of gravity to the surface of the fluid. The same function likewise determines the figure of the mass; for, the fluid being at liberty, the surface will contain all the points at which there is no pressure. If  $p$  denote the pressure at any interior point  $(x\ y\ z)$ , this equation has been investigated, viz.

$$C - p = \phi'(x, y, z);$$

and if we make  $p = 0$ , the result, viz.

$$C = \phi'(x, y, z)$$

must be verified at all the points of the surface. But it is to be observed, that instances may occur in which the function  $\phi'(x, y, z)$  in passing from a point within the fluid to a point in the surface, undergoes a modification in the form of its expression. It may happen that the quantities which it contains acquire particular relations at the surface; and on this account the function may put on a *singular form*, distinguished

\* If the mathematical principle of the property respecting the canals be stated abstractly, it will be found to lie in the nature of the function  $p$ , which must be a maximum at the centre of gravity, the point of greatest pressure; and continually decreasing in receding from that point, it must have the same value at all points of the surface of the fluid. Now it is not impossible but, in some problems, there may be more than one function that will satisfy the two conditions; and, should this be the case, the figure of the fluid remaining the same, the property respecting the canals would be verified in more than one supposition respecting the pressure and the forces in action.

in some respects from the original expression as it exists in the interior parts. We may suppose that  $\phi' (x, y, z)$  changes into  $\phi (x, y, z)$  at the surface of the fluid; inso-much that  $\phi' (x, y, z)$  and  $\phi (x, y, z)$  are identical for all the points in the surface, but are different from one another when the coordinates of any other point are substituted. The pressure at any interior point being determined by the expression

$$C - p = \phi' (x, y, z);$$

the equation of the fluid's surface will be

$$C = \phi (x, y, z);$$

the first formula being identical with the second at the surface, or when  $p = 0$ .

The hypothesis of which we have been speaking is not an imaginary one, for a homogeneous planet in a fluid state is an example in point. In this case the forces in action are partly the attraction of the mass upon a particle; and as the fluid has a spheroidal form, the attraction upon a particle in the surface is more simple in its expression, and depends upon fewer quantities than the like force upon a point within the surface. Although it is true universally that the forces urging a particle in the surface of a fluid in equilibrium are deducible from the general expressions of the forces in the interior parts, yet in such cases as that mentioned it does not hold conversely that the latter forces are deducible from the former. This distinction, which is important, is not attended to in CLAIRAUT's theory, which tacitly assumes that the forces are invariably expressed by the same functions without any change of form, whether the point of action be in or below the surface of the fluid.

It appears from what has been said, that in solving problems of equilibrium it is necessary to begin with inquiring in what manner the forces at the surface, which always depend upon the equation of the surface, are connected with the forces supposed to act upon the particles within the surface. A twofold division must be distinguished. In the first and more simple class of problems, it is assumed that the function  $\phi' (x, y, z)$  from which the forces are deduced, undergoes no modification at the surface, but retains immutably the same form of expression at every point of the mass. In the other class of problems the function  $\phi' (x, y, z)$  is supposed to undergo some modification at the surface of the fluid; so that the forces in the interior parts admit of a twofold expression, one derived from the original function  $\phi' (x, y, z)$ , and another from the particular form  $\phi (x, y, z)$ , which that function assumes at the surface. In such cases the equilibrium will depend upon two different algebraic expressions, and not upon one only, as in the first division, or in CLAIRAUT's theory.

5. The following theorem contains all that concerns the equilibrium in the first and more simple hypothesis, namely, when the functions of the coordinates which express the forces undergo no change of form in passing from a point in the surface of the fluid to a point within the surface.

*Theorem.*

If a body of homogeneous fluid at liberty have for the equation of its surface,

$$C = \phi(x, y, z),$$

the mass will be in equilibrium, supposing that every particle  $(x, y, z)$  is urged by the forces  $X, Y, Z$ , respectively parallel to the rectangular coordinates  $x, y, z$ , and equal to the partial differential coefficients of  $\phi(x, y, z)$ , that is,

$$X = \frac{d \cdot \phi(x, y, z)}{d x}, \quad Y = \frac{d \cdot \phi(x, y, z)}{d y}, \quad Z = \frac{d \cdot \phi(x, y, z)}{d z}.$$

The origin of the coordinates being placed at the centre of gravity of the mass, the theorem must be supposed to assume further, that the expressions of the forces, that is, the differential coefficients of  $\phi(x, y, z)$ , vanish when the coordinates are all equal to zero; for without this condition the equilibrium of the mass of fluid would be impossible. From this it follows that the value of  $\phi(x, y, z)$  will increase continually as the point  $(x, y, z)$  recedes from the centre and approaches the surface of the fluid on any side. If  $C^\circ$  denote the value of  $\phi(x, y, z)$  at the centre of gravity,  $C - C^\circ$  will be the whole increase in varying from that point to the surface of the fluid; and as every gradation of magnitude is passed through, an interior surface may be found that will satisfy the equation

$$C' = \phi(x, y, z),$$

provided  $C'$  be any quantity between the limits  $C$  and  $C^\circ$ . Wherefore if  $C - C^\circ$  be divided into an infinite number of elementary portions, each equal to  $\delta p$ , there will exist a series of curve surfaces gradually contracting in their dimensions round the centre, and the last containing a drop of fluid, which may be as small as we please; of which successive curve surfaces, beginning with the upper surface of the mass, these are the respective equations:

$$C = \phi(x, y, z), \text{ or } C = \phi,$$

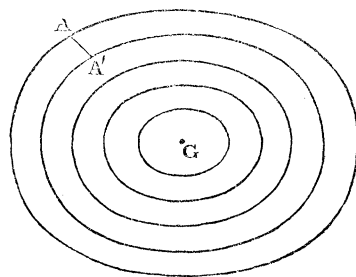
$$C - \delta p = \phi,$$

$$C - 2 \delta p = \phi,$$

$$C - 3 \delta p = \phi, \text{ \&c.}$$

From  $A$  in the upper surface draw  $AA'$  perpendicular to the surface next below; put  $k = AA'$ , the thickness of the stratum; and let  $w$  denote any infinitely small portion of the curve surface at  $A'$ ; then  $k \times w$  will be the portion of the stratum insisting on the small surface  $w$ . The coordinates of the point  $A'$  being  $x, y, z$ , the forces in action and respectively parallel to the coordinates will be

$$\frac{d \phi}{d x}, \quad \frac{d \phi}{d y}, \quad \frac{d \phi}{d z};$$



and by these forces we may suppose that every particle in the small mass  $k \times w$  is urged. Now let

$$F = \sqrt{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2};$$

and by a well-known property, the cosines of the angles which the coordinates of the lower surface of the stratum make with the normal of the same surface, will be respectively equal to

$$\frac{1}{F} \times \frac{d\phi}{dx}, \quad \frac{1}{F} \times \frac{d\phi}{dy}, \quad \frac{1}{F} \times \frac{d\phi}{dz};$$

and hence the sum of the partial forces acting in the direction of  $k$ , will be equal to

$$\frac{1}{F} \cdot \left(\frac{d\phi}{dx}\right)^2 + \frac{1}{F} \cdot \left(\frac{d\phi}{dy}\right)^2 + \frac{1}{F} \cdot \left(\frac{d\phi}{dz}\right)^2 = F;$$

wherefore,  $F \times k \times w$  will be the impulse or pressure exerted by the small mass  $k \times w$  upon the small surface  $w$ , on which it insists. Again, the coordinates of the end of  $k$  in the upper surface of the stratum, are

$$x + \frac{k}{F} \cdot \frac{d\phi}{dx}, \quad y + \frac{k}{F} \cdot \frac{d\phi}{dy}, \quad z + \frac{k}{F} \cdot \frac{d\phi}{dz};$$

and as the equations of the two curve surfaces are

$$C = \phi \cdot \left(x + \frac{k}{F} \cdot \frac{d\phi}{dx}, \quad y + \frac{k}{F} \cdot \frac{d\phi}{dy}, \quad z + \frac{k}{F} \cdot \frac{d\phi}{dz}\right),$$

$$C - \delta p = \phi \cdot (x, y, z);$$

we deduce,

$$\delta p = \frac{k}{F} \cdot \left\{ \left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2 \right\} = k \times F.$$

Wherefore, the pressure  $F \times k \times w$  of the mass  $k \times w$  upon the small surface  $w$ , will be equal to  $\delta p \times w$ ; which proves that the incumbent stratum exerts a constant pressure upon the surface passing through  $A'$ , the intensity at every point being equal to  $\delta p$ . The like demonstration may be employed to show, that any other stratum exerts a constant pressure upon the fluid below it; and hence it follows, that all the fluid above any of the interior surfaces, whatever be the number of strata it consists of, presses with the same intensity at every point of the surface. Now the forces urging the particles of the fluid decrease continually in approaching the centre of gravity, at which point they are evanescent: wherefore the infinitely small mass, or drop, contained within the surface nearest the centre, may be considered as free from the action of any accelerating forces; and, its surface being subjected to the constant pressure of all the incumbent strata, these pressures, the directions of which ultimately pass through the centre of gravity, will balance one another without any tendency to produce either progressive or rotatory motion.

If  $n$  be the number of strata above any of the interior surfaces, the intensity of

pressure at all the points of the surface will be  $n \times \delta p$ ; and the equation of the surface being

$$C - n \times \delta p = \varphi(x, y, z),$$

if  $p = n \times \delta p$ , we shall have

$$C - p = \varphi(x, y, z),$$

which equation ascertains the pressure at any point  $(x y z)$ , and determines the surface containing all the points at which the same pressure prevails. This agrees with what was investigated in No. 3.

The interior surfaces at all the points of which the pressure is constant have been called *level surfaces*; and a stratum of the fluid lying between two level surfaces is called a *level stratum*.

A property common to all the level surfaces, and to the upper surface of the fluid, consists in this, that the resultant of the forces acting upon a particle contained in any of these surfaces is directed perpendicularly towards it. Take two points,  $(x y z)$  and  $(x + \delta x, y + \delta y, z + \delta z)$ , infinitely near one another in the surface of which the equation is

$$C - p = \varphi(x, y, z);$$

and put  $\delta s$  for the short line between the two points; by differentiating,  $C - p$  being constant, we get

$$\frac{d\varphi}{dx} \cdot \frac{dx}{ds} + \frac{d\varphi}{dy} \cdot \frac{dy}{ds} + \frac{d\varphi}{dz} \cdot \frac{dz}{ds} = 0;$$

or, which is equivalent,

$$X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} = 0:$$

Now  $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$  are the cosines of the angles which the directions of the forces make with the line  $ds$ : wherefore the algebraic expression in the last equation is the sum of the partial forces which act in the direction of  $ds$ ; and as this sum is equal to zero in all positions of that line round the point  $(x y z)$ , the forces will produce no effect in the plane touching the curve surface, and will exert their whole action at right angles to the surface.

From what is here investigated, we may derive this general property: If the forces  $X, Y, Z$ , which vary from point to point, be always perpendicular to a surface, they must satisfy this equation,

$$X dx + Y dy + Z dz = 0,$$

the coordinates being made to vary in the surface. For if the equation be divided by  $ds = \sqrt{dx^2 + dy^2 + dz^2}$ , the result will be

$$X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dz}{ds} = 0,$$

which expresses, as is shown above, that the whole action of the forces is perpendicular to the surface.

It may be observed further with respect to the level surfaces, that in forming their equations, nothing is supposed to change in the general equation

$$C - p = \varphi(x, y, z),$$

except the quantity  $C - p$ , which is constant in every individual surface, and the values of the coordinates, the form of the function  $\varphi(x, y, z)$ , and all the coefficients it contains, remaining immutably fixed. Every particular surface has, therefore, its independent equation, which is completely defined when the value of its constant is ascertained: and, as the equation of the upper surface determines the equilibrium of the whole mass of fluid, so, for the very same reasons, the equation of any interior level surface will determine the separate equilibrium of the fluid within it, supposing the constant pressure of the incumbent stratum to be taken off or annihilated.

The foregoing theorem, which is equivalent to the theory of CLAIRAUT, cannot possibly be attended with any difficulty. But if the simplicity of the matter conduces to make it clear, it also greatly narrows its application. The theorem is sufficient for determining the equilibrium when the forces are explicit functions of the coordinates of the point of action; that is, such functions as are entirely known when the values of the coordinates are assigned. In this case, the differential equation of the surface must first be formed; and, this being integrated, we obtain the equation of the figure which the fluid must assume.

But the theorem is not sufficient for determining the equilibrium when a fluid consists of particles that mutually attract one another; because, in this case, the forces, varying with the figure of the fluid, are not explicit functions of the coordinates of the point of action; and because the expressions of the forces for a point in the surface of the fluid are in some respects different from the like expressions for a point within the surface, which is contrary to the hypothesis of the theorem. The problem thus assumes a new aspect, and further researches are required for its solution.

6. In the second division of problems, if the equation of the surface of a mass of fluid be

$$C = \varphi(x, y, z) \text{ or } C = \varphi,$$

the forces which urge the particles within the surface are expressed by the differential coefficients, viz.

$$\frac{d\varphi'}{dx}, \quad \frac{d\varphi'}{dy}, \quad \frac{d\varphi'}{dz},$$

of a function  $\varphi'(x, y, z)$ , which is different from  $\varphi(x, y, z)$ , for all the points within the surface, and identical with it for the points in the surface. The equilibrium requires that the forces acting upon the interior particles, or the differentials of  $\varphi'(x, y, z)$ , vanish at the origin of the coordinates in the centre of gravity; and this will not take place if  $\varphi'(x, y, z)$  contain any terms such as  $Ax$ ,  $By$ ,  $Cz$ , the coefficients  $A$ ,  $B$ ,  $C$  being constant quantities. And since  $\varphi'(x, y, z)$  is changed into  $\varphi(x, y, z)$  when the

coordinates have particular values, it follows that  $\phi(x, y, z)$  will contain no terms such as  $Ax, By, Cz$ ; and consequently that its differential coefficients, viz.

$$\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz},$$

will vanish at the centre of gravity. Wherefore, in all problems of this class, the foregoing theorem may be applied to the equation of the surface of the fluid, since the necessary conditions are fulfilled.

Now attending solely to the equation of the surface, viz.

$$C = \phi(x, y, z),$$

it has been shown that the expressions

$$\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz},$$

represent forces respectively parallel to the coordinates, the resultant of which is directed perpendicularly towards the surface. If it be supposed that every particle of the fluid is urged by forces expressed by substituting its coordinates instead of the coordinates of the surface in the same functions  $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$ , it is proved in the theorem that the mass will be in equilibrium, and may be divided by an infinite number of level surfaces into thin strata that exert a constant pressure upon one another. We have, therefore, now to inquire how the equilibrium which takes place when  $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$  are the forces in action, is to be preserved when, instead of these, the other forces,  $\frac{d\phi'}{dx}, \frac{d\phi'}{dy}, \frac{d\phi'}{dz}$ , are substituted. These latter forces may be considered as produced by additions made to the first, and they may be thus written,

$$\frac{d\phi}{dx} + \left(\frac{d\phi'}{dx} - \frac{d\phi}{dx}\right), \frac{d\phi}{dy} + \left(\frac{d\phi'}{dy} - \frac{d\phi}{dy}\right), \frac{d\phi}{dz} + \left(\frac{d\phi'}{dz} - \frac{d\phi}{dz}\right):$$

and supposing the whole body of fluid to be divided, as in the theorem, into thin level strata, to which the joint action of the forces  $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$  is at every point perpendicular, it is evident that the equilibrium will be destroyed when the additional forces come into action, unless their resultant, urging any particle, be perpendicular to the level surface in which the particle is contained; but if the resultant be perpendicular to the level surface, the equilibrium will not be disturbed, because the thin strata will still continue to exert a constant pressure upon one another in like manner as before the new forces were introduced \*. However the additional forces

$$\left(\frac{d\phi'}{dx} - \frac{d\phi}{dx}\right), \left(\frac{d\phi'}{dy} - \frac{d\phi}{dy}\right), \left(\frac{d\phi'}{dz} - \frac{d\phi}{dz}\right)$$

\* It is by means of this very general principle that we pass from the equilibrium of a homogeneous fluid to that of one in which the density, being constant at all the points of the same level surface, varies, according to any law, from one level surface to another.

be supposed to vary in passing from one level surface to another, there will be no tendency to destroy the equilibrium, provided their action be directed perpendicularly to every such surface. The perpendicularity of the resultant of the additional forces to a level surface is expressed by this equation,

$$\left(\frac{d\phi'}{dx} - \frac{d\phi}{dx}\right) dx + \left(\frac{d\phi'}{dy} - \frac{d\phi}{dy}\right) dy + \left(\frac{d\phi'}{dz} - \frac{d\phi}{dz}\right) dz = 0;$$

or more simply by this,

$$d \cdot \phi' (x, y, z) - d \cdot \phi (x, y, z) = 0,$$

the coordinates varying in the level surface.

We can now assign the conditions necessary for the equilibrium of a mass of homogeneous fluid at liberty, the particles of which are urged by the forces  $\frac{d\phi}{dx}$ ,  $\frac{d\phi}{dy}$ ,  $\frac{d\phi}{dz}$ , at the surface, and by the forces  $\frac{d\phi'}{dx}$ ,  $\frac{d\phi'}{dy}$ ,  $\frac{d\phi'}{dz}$ , within the surface; the functions  $\phi(x, y, z)$  and  $\phi'(x, y, z)$  being identical for all the points in the surface, and different from one another for all the points within the surface: first, the resultant of the forces in action at the surface must be directed perpendicularly towards that surface; and secondly, supposing the coordinates to vary from point to point of the same level surface, the differential equation

$$d \cdot \phi' (x, y, z) - d \cdot \phi (x, y, z) = 0$$

must be verified at all the points of the level surface.

In the hypothesis respecting the forces under consideration, there are two independent pressures at every interior point of the fluid; one caused by the forces  $\frac{d\phi}{dx}$ ,  $\frac{d\phi}{dy}$ ,  $\frac{d\phi}{dz}$ , deduced from the equation of the upper surface of the fluid; and the other by the additional forces

$$\left(\frac{d\phi'}{dx} - \frac{d\phi}{dx}\right), \left(\frac{d\phi'}{dy} - \frac{d\phi}{dy}\right), \left(\frac{d\phi'}{dz} - \frac{d\phi}{dz}\right):$$

and the equilibrium of the fluid will be impossible unless the mass can be partitioned by an infinite number of surfaces, in every one of which the two pressures are both constant\*. Now the pressure caused by the forces  $\frac{d\phi}{dx}$ ,  $\frac{d\phi}{dy}$ ,  $\frac{d\phi}{dz}$ , is constant in all the surfaces called level surfaces in the theorem; and as these surfaces depend solely on the equation of the figure of the fluid, it is obvious that no figure can be induced on the mass that will secure the equilibrium, unless the pressure caused by the additional forces be likewise perpendicular to the same level surfaces. But if both pressures be constant at all the points of every level surface, which is the condition expressed by the equation

$$d \cdot \phi' (x, y, z) - d \cdot \phi (x, y, z) = 0,$$

\* In no other way is it possible that the pressures propagated through the mass can balance and sustain one another.

the equilibrium of the fluid will obviously be a consequence of the theorem. It is therefore demonstrated, with respect to problems of the second class, that the equation of the upper surface of the fluid is not sufficient by itself to determine the equilibrium of the mass.

In the theorem, the term level surface is liable to no ambiguity; but in the more complex state of the forces that occurs in the second division of problems, two different systems of surfaces in which the pressure is constant require attention; for the pressure caused by the forces  $\frac{d\phi}{dx}$ ,  $\frac{d\phi}{dy}$ ,  $\frac{d\phi}{dz}$ , is constant in all interior surfaces determined by the equation

$$d \cdot \phi (x, y, z) = 0;$$

and the pressure caused by the forces  $\frac{d\phi'}{dx}$ ,  $\frac{d\phi'}{dy}$ ,  $\frac{d\phi'}{dz}$ , is constant in all surfaces of which the general equation is

$$d \cdot \phi' (x, y, z) = 0.$$

It will therefore conduce to clearness if the meaning of a level surface be restricted, by adding to the two properties of being perpendicular to the resultant of the forces acting on the particles contained in it, and being pressed at all its points with the same intensity, the further condition of being deduced by varying the constant in the equation of the upper surface of the fluid. The effect of the equation

$$d \cdot \phi' (x, y, z) - d \cdot \phi (x, y, z) = 0$$

is to verify the two differential equations above mentioned at all the points of the same surface: it implies that the two systems of surfaces of constant pressure are blended in one; and as this is a necessary condition of equilibrium, it distinguishes from all other figures those which are alone susceptible of an equilibrium.

7. The general theory of the equilibrium of homogeneous fluids at liberty having been explained at sufficient length, it is next to be applied to some of the principal problems.

#### PROBLEM I.

To determine the equilibrium of a homogeneous fluid at liberty, the particles attracting one another with a force inversely proportional to the square of the distance, at the same time that they are urged by a centrifugal force caused by revolving about an axis.

The mass of fluid being in equilibrium, the centre of gravity will be free from the action of any forces; and as the attractive forces balance one another at that point, there must be no centrifugal force at the same point; that is, the axis of rotation must pass through it.

The origin of the coordinates being placed in the centre of gravity, let  $x, y, z$ , denote the rectangular coordinates of a particle of the fluid, and  $x', y', z'$ , those of a molecule  $dm$  of the mass, the two coordinates  $x$  and  $x'$  being parallel to the axis of

rotation: and  $f$  being the distance between the assumed particle and the molecule  $d m$ , we shall have

$$f = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

The attraction of spheres, according to the law supposed in this problem, being the same as if all the matter were collected in the centre, we may adopt for the unit of mass a sphere of the fluid having its radius equal to the unit of distance; and for the unit of force, the attraction of the sphere upon a point in its surface; then the direct attraction of the molecule  $d m$  upon the particle at the distance  $f$ , will be  $\frac{d m}{f^2}$ ; and the partial attractions urging the particle inward in the directions of  $x, y, z$ , will be respectively equal to

$$\frac{d m}{f^2} \cdot \frac{x - x'}{f}, \quad \frac{d m}{f^2} \cdot \frac{y - y'}{f}, \quad \frac{d m}{f^2} \cdot \frac{z - z'}{f}.$$

Now if we observe that

$$\frac{x - x'}{f} = \frac{d f}{d x}, \quad \frac{y - y'}{f} = \frac{d f}{d y}, \quad \frac{z - z'}{f} = \frac{d f}{d z},$$

it will readily appear that the sums of the attractive forces, with which all the molecules of the mass urge the particle inward in the respective directions of  $x, y, z$ , may be thus commodiously expressed:

$$- \frac{d \cdot \int \frac{d m}{f}}{d x}, \quad - \frac{d \cdot \int \frac{d m}{f}}{d y}, \quad - \frac{d \cdot \int \frac{d m}{f}}{d z},$$

the integral extending to all the molecules of the mass.

The attraction of the sphere at its surface being represented by unit, the velocity communicated by that force in the infinitely short time  $d t$ , will be  $1 \times d t$ ; and if the time of one entire revolution about the axis of rotation be denoted by  $T$ , the velocity generated by the centrifugal force at the distance of unit from the axis in the time  $d t$ , will be  $\frac{4 \pi^2}{T^2} \times d t$ ; wherefore the centrifugal force acting at the distance of unit from the axis of rotation, and estimated in parts of the unit of force, will be equal to

$$\frac{4 \pi^2}{T^2} = \varepsilon.$$

At the distance of  $\sqrt{y^2 + z^2}$  from the axis, the centrifugal force will therefore be  $\varepsilon \times \sqrt{y^2 + z^2}$ ; and the resolved parts of it which urge the particle in the prolongations of  $y$  and  $z$ , will be equal to  $\varepsilon \times y$  and  $\varepsilon \times z$ .

Now if  $X, Y, Z$  represent the total forces tending inward and urging the assumed particle in the directions of the coordinates, we shall have

$$X = - \frac{d \cdot \int \frac{d m}{f}}{d x}, \quad Y = - \left( \frac{d \cdot \int \frac{d m}{f}}{d y} + \varepsilon y \right), \quad Z = - \left( \frac{d \cdot \int \frac{d m}{f}}{d z} + \varepsilon z \right);$$



rection of  $R$ : then  $\left[\int \frac{d m}{f}\right]$  being a function of two dimensions, in which  $x, y, z$  are the only variables,  $\frac{1}{R^2} \times \left[\int \frac{d m}{f}\right]$  will be a quantity of no dimensions; it will, therefore, be a function of  $\frac{x}{R}, \frac{y}{R}, \frac{z}{R}$ , or of  $a, b, c$ ; so that we shall have

$$\left[\int \frac{d m}{f}\right] = R^2 \times F(a, b, c), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$F$  being the mark of a function. The same value may be expressed by means of the coordinates, viz.

$$\left[\int \frac{d m}{f}\right] = (x^2 + y^2 + z^2) \times F\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

If the value just found be substituted in equation (2.), the result will be,

$$C = (x^2 + y^2 + z^2) \times F\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) + \frac{\varepsilon}{2} \times (y^2 + z^2):$$

which proves that the forces in action at the surface of the fluid are not sufficient to determine the equilibrium of the mass. For the equation of the figure of the fluid at which we have arrived, containing an arbitrary function, is indeterminate; and, on examination, it will be found to comprehend the ellipsoid and innumerable other figures\*.

If for  $x, y, z$  we substitute their values  $R a, R b, R c$ , the equation of the surface will assume this form,

$$C = R^2 \times \left\{ F(a, b, c) + \frac{\varepsilon}{2} (b^2 + c^2) \right\}.$$

The equation of a level surface is deduced from the equation of the upper surface by changing the constant, and substituting the coordinates of the level surface for those of the upper surface: now, supposing that  $r$ , in the same straight line with  $R$ , is a radius of a level surface, the coordinates of the point in that surface at the extremity of  $r$  will be  $r a, r b, r c$ , because  $r$  and  $R$  have the same direction: wherefore, by substituting  $r a, r b, r c$  for  $x, y, z$  in the equation of the upper surface, and denoting the new constant by  $C'$ , the equation of the level surface of which  $r$  is the radius will be

$$C' = r^2 \times \left\{ F(a, b, c) + \frac{\varepsilon}{2} (b^2 + c^2) \right\}.$$

The comparison of this equation with that of the upper surface of the fluid leads to this result,

$$\frac{r^2}{R^2} = \frac{C'}{C};$$

\* In a particular examination of CLAIRAUT's theory that occurs in the sequel of this Paper, it is proved from different principles, that the equation of the figure of the fluid deduced from the forces in action at the surface is indeterminate, and admits of innumerable solutions.

from which it follows, that every interior level surface is similar to the upper surface, and similarly posited about the centre of gravity.

The expression of the integral in equation (3.) is evidently true in all similar spheroids, without any change in the function  $F$ ; for  $F$ , being of no dimensions, contains only the proportions of the linear dimensions of the geometrical figures, and these proportions are the same when the figures are similar. And, since all the level surfaces are similar to the upper surface, it is obvious that the equation of a level surface may be thus expressed,

$$C' = \left[ \int \frac{dm}{f} \right] + \frac{\varepsilon}{2} \cdot (y^2 + z^2) :$$

because the integral between the brackets, which stands for the sum of all the molecules within the level surface divided by their respective distances from a point  $(xyz)$  in that surface, is equal to the part of the equation of the level surface which contains the function  $F$ . Now the equilibrium of the mass of fluid will be impossible, unless the pressure determined by the equation (1.) be constant at all the points of the same level surface; which requires that the equation

$$d \cdot \left\{ \int \frac{dm}{f} + \frac{\varepsilon}{2} (y^2 + z^2) \right\} = d \cdot \left\{ \left[ \int \frac{dm}{f} \right] + \varepsilon (y^2 + z^2) \right\}$$

be verified, the coordinates of the attracted point varying in any level surface\*. This differential equation will be fulfilled if the equation

$$\text{constant} = \int \frac{dm}{f} - \left[ \int \frac{dm}{f} \right]$$

hold at all the points of every level surface. And as the integral without brackets is the sum of all the molecules of the whole mass of fluid, divided by their respective distances from the attracted point in the level surface; and the integral with brackets is the like sum relatively to all the molecules within the level surface; the last equation may be expressed more simply thus,

$$\text{constant} = \int \frac{dm}{f},$$

the integral being extended to all the molecules of the stratum between the level surface and the upper surface of the fluid. In the figures which verify this equation there will exist in the interior parts no surfaces of constant pressure except the level surfaces, which is a necessary condition of equilibrium; and the intensity of pressure in every level surface will be determined by the equation (1.), as required in the problem.

We have next to investigate the figures which verify this last equation. Let  $s$  represent the distance of  $dm$  from the centre of gravity; and,  $r$  being drawn to an

\* That is, of every surface determined by varying the constant in the equation of the upper surface, according to the definition in No. 6.

attracted point in a level surface, put  $\theta$  and  $\theta'$  for the angles which  $r$  and  $s$  make with the axis of rotation; and  $\varpi$  and  $\varpi'$  for the angles which determine the positions of the projections of  $r$  and  $s$  upon a plane passing through the centre of gravity perpendicular to the same axis: then  $\psi$  being the angle between the two lines  $r$  and  $s$ , and  $f$  the distance of  $dm$  from the attracted point, we shall have

$$\gamma = \cos \psi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varpi - \varpi'),$$

$$f = \sqrt{s^2 - 2sr \cdot \gamma + r^2}.$$

Again, if the plane of the two lines  $r$  and  $s$  describe the small angle  $d\sigma$  by revolving about  $r$ , the extremity of  $s$  will describe the short line  $s \cos \psi d\sigma$  perpendicular to the revolving plane: further, supposing that the arc  $\psi$  increases to  $\psi + d\psi$ , the extremity of  $s$  will move through the short line  $s d\psi$  in the plane of the arc  $\psi$ ; now the short lines  $s \cos \psi d\sigma$  and  $s d\psi$  being perpendicular to one another and to  $s$ , the molecule  $dm$  may be considered equal to  $s \cos \psi d\sigma \times s d\psi \times ds$ ; or, which is the same thing, we may assume

$$dm = -d\gamma d\sigma \cdot s^2 ds.$$

By substituting the values of  $dm$  and  $f$ , the integral under consideration will be thus expressed:

$$\int \frac{dm}{f} = \iint -d\gamma d\sigma \int \frac{s^2 ds}{\sqrt{s^2 - 2sr \cdot \gamma + r^2}},$$

the integrations being extended to all values from  $\gamma = 1, \sigma = 0$ , to  $\gamma = -1, \sigma = 2\pi$ , and from  $s = r'$ , to  $s = R'$ ,  $r'$  and  $R'$  being two radii in the same straight line, the first of a level surface, and the other of the upper surface of the fluid. The radical quantity must now be expanded in a series of the powers of  $\frac{r}{s}$ , viz.

$$\frac{1}{s} + \frac{r}{s} \cdot C^{(1)} + \frac{r^2}{s^3} \cdot C^{(2)} + \frac{r^3}{s^5} \cdot C^{(3)} +, \&c.,$$

the coefficients being determined by the formula

$$C^{(i)} = \frac{1}{2^i} \times \frac{d^i (\gamma^2 - 1)^i}{1 \cdot 2 \cdot 3 \dots i d\gamma^i} : *$$

and having substituted this series, and effected the integrations with respect to  $ds$  between the assigned limits, the result will be

$$\begin{aligned} \int \frac{dm}{f} &= \frac{1}{2} \iint -d\gamma d\sigma (R'^2 - r'^2) \\ &+ r \iint -d\gamma d\sigma (R' - r') C^{(1)} \\ &+ r^2 \iint -d\gamma d\sigma \log \frac{R'}{r'} \times C^{(2)} \end{aligned}$$

\* Vide Appendix.

$$\begin{aligned}
& + r^3 \iint -d\gamma d\sigma \left( \frac{1}{r'} - \frac{1}{R'} \right) C^{(3)} \\
& + \frac{r^4}{2} \iint -d\gamma d\sigma \left( \frac{1}{r'^2} - \frac{1}{R'^2} \right) C^{(4)} \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \frac{r^i}{i-2} \iint -d\gamma d\sigma \left( \frac{1}{r'^{i-2}} - \frac{1}{R'^{i-2}} \right) C^{(i)}.
\end{aligned}$$

Because every level surface is similar to the upper surface, and similarly posited about the centre of gravity, and that  $r$  and  $R$ , as well as  $r'$  and  $R'$ , are radii of the two surfaces in the same straight line, we have

$$r = \alpha \cdot R, \quad r' = \alpha \cdot R',$$

$\alpha$  being a fraction of unit, which is the same for all the points of the same level surface; wherefore, by substituting the values of  $r$  and  $r'$ , and leaving out the term

$$r^2 \iint -d\gamma d\sigma \log \frac{R'}{r'} \times C^{(2)} = R^2 \alpha^2 \log \frac{1}{\alpha} \cdot \iint -d\gamma d\sigma C^{(2)},$$

which is equal to zero, we get

$$\begin{aligned}
\int \frac{dm}{f} &= \frac{1-\alpha^2}{2} \iint -d\gamma d\sigma \cdot R'^2 \\
&+ (\alpha - \alpha^2) R \iint -d\gamma d\sigma C^{(1)} \cdot R' \\
&+ (\alpha^2 - \alpha^3) R^3 \iint \frac{-d\gamma d\sigma C^{(3)}}{R'} \\
&- \frac{(\alpha^3 - \alpha^4) R^4}{2} \cdot \iint \frac{-d\gamma d\sigma C^{(4)}}{R'^2} \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\frac{(\alpha^2 - \alpha^i) R^i}{i-2} \iint \frac{-d\gamma d\sigma C^{(i)}}{R'^{i-2}}.
\end{aligned}$$

Such is the expression of the integral under consideration, the attracted point being the intersection of  $R$ , with the level surface of which  $\alpha R$  is a radius; and the value of the integral must be constant at all the points of the same level surface, that is, it must be the same when  $\alpha$  is the same in whatever direction  $R$  be drawn.

In the first place, if the figure of the fluid be a sphere, the expression of  $\int \frac{dm}{f}$  is reduced to its first term, which is constant in every spherical surface concentric with

the upper surface; because by the nature of the functions  $C^{(i)}$  all the integrals vanish when  $R'$  is constant. But the supposition of a sphere requires that  $\varepsilon$  be equal to zero in the equations (1.) and (2.), or that there be no centrifugal force.

But if the radius  $R$  vary as it changes its direction,  $\int \frac{d m}{f}$  cannot be of the same quantity at every point of the same level surface, except when all the terms after the first are separately equal to zero, that is, except the expression of  $R'$  be such that

$$\iint \frac{-d\gamma d\sigma C^{(i)}}{R^{i-2}} = 0,$$

for all values of  $i$  from 1 to  $\infty$ .

The investigation will be greatly facilitated by the following theorem:

If  $a' = \cos \vartheta'$ ,  $b' = \sin \vartheta' \cos \varpi'$ ,  $c' = \sin \vartheta' \sin \varpi'$ , the integral

$$\iint -d\gamma d\sigma C^{(i)} a'^m b'^{m'} c'^{m''},$$

extended to all values of  $\gamma$  from 1 to  $-1$ , and of  $\sigma$  from 0 to  $2\pi$ , will be equal to zero in all cases when  $m + m' + m''$  is less than  $i^*$ .

It is obvious that  $\frac{1}{R^{i/2}}$  is a function of the three quantities  $a'$ ,  $b'$ ,  $c'$ ; and if we assume

$$\frac{1}{R^{i/2}} = U^{(0)} + U^{(1)} + U^{(2)},$$

$U^{(0)}$  being a constant, and  $U^{(1)}$ ,  $U^{(2)}$  functions such that  $a'$ ,  $b'$ ,  $c'$  rise to one dimension in all the terms of  $U^{(1)}$ , and to two dimensions in all the terms of  $U^{(2)}$ , the highest sum of the indexes in the combinations of  $a'$ ,  $b'$ ,  $c'$ , contained in the expressions of  $\frac{1}{R^4}$ ,  $\frac{1}{R^6}$ ,  $\frac{1}{R^8}$ , &c., will not exceed 4, 6, 8, &c.: wherefore, by the theorem, the assumed value of  $\frac{1}{R^{i/2}}$  will succeed in making all the terms of  $\int \frac{d m}{f}$  vanish in which  $i$  is an even number, and it is evidently the most general assumption for  $\frac{1}{R^{i/2}}$  that will answer the same end.

When  $i$  is an odd number, we have

$$\iint \frac{-d\gamma d\sigma C^{(i)}}{R^{i-2}} = \iint \frac{-d\gamma d\sigma C^{(i)}}{(U^{(0)} + U^{(1)} + U^{(2)})^{\frac{i-2}{2}}}.$$

In this case  $C^{(i)}$ , being an odd function of  $\gamma$ , is the same in quantity, but changes its sign, when for  $\vartheta'$  and  $\varpi'$  we substitute  $\vartheta' + \frac{\pi}{2}$  and  $\varpi' + \pi$ : wherefore the whole integral will be equal to zero, if the denominator retain the same positive value when  $\vartheta'$  and  $\varpi'$  are changed into  $\vartheta' + \frac{\pi}{2}$  and  $\varpi' + \pi$ , the increase of the integral being, on

\* Vide Appendix.

this supposition, exactly compensated by the decrease. But this requires that  $U^{(1)}$  be exterminated, because this function varies its sign when  $\theta'$  and  $\varpi'$  are changed into  $\theta' + \frac{\pi}{2}$  and  $\varpi' + \pi$ . Wherefore, leaving out  $U^{(1)}$ , if we assume

$$\frac{1}{R'^2} = U^{(0)} + U^{(2)},$$

it will follow from what has been said, that all the terms of  $\int \frac{d m}{f}$  after the first, both those in which  $i$  is even and those in which it is odd, will vanish, so that we shall have

$$\int \frac{d m}{f} = \frac{1 - \alpha^2}{2} \iint - d \gamma d \sigma R'^2,$$

which is constantly of the same value at all the points of the same level surface.

Taking the most general expression of  $U^{(2)}$ , and observing that the constant

$$U^{(0)} = U^{(0)} a'^2 + U^{(0)} b'^2 + U^{(0)} c'^2,$$

may be blended with  $U^{(2)}$ , we shall have

$$\frac{1}{R'^2} = A a'^2 + B b'^2 + C c'^2 + D a' b' + E a' c' + F b' c':$$

but  $x', y', z'$  being the coordinates of  $R'$  in the surface of the fluid, we have

$$\frac{x'}{R'} = a', \quad \frac{y'}{R'} = b', \quad \frac{z'}{R'} = c':$$

and these values being substituted, the result will be

$$1 = A x'^2 + B y'^2 + C z'^2 + D x' y' + E x' z' + F y' z',$$

which is the equation of an ellipsoid, the coordinates  $x', y', z'$  being parallel to three diameters intersecting at right angles. It is therefore demonstrated, that the ellipsoid comprehends all the figures that will make the integral  $\int \frac{d m}{f}$ , taken between the assigned limits, of the same value at all the points of the same level surface, that is, at all the points of any interior surface similar to the upper surface, and similarly posited about the centre.

The foregoing reasoning is independent of the centrifugal force; but by attending to the rotatory motion which causes that force, it is easy to prove that the axis about which the fluid revolves, or the diameter parallel to the coordinate  $x'$ , must coincide with one of the axes of the geometrical figure. For, there being no centrifugal force at the poles of the axis of rotation in the surface of the fluid, the only force in action at these points is the attraction of the mass. But the resultant of the forces urging every particle in the surface of a fluid in equilibrium must be perpendicular to the surface: and as there are no points on the surface of an ellipsoid at which the attraction of the mass is perpendicular to the surface, except the extremities of the three axes, it follows that, with one or other of these, the axis of rotation of the fluid in

equilibrium must coincide. The diameter parallel to  $x'$  being thus proved to be an axis of the ellipsoid, we may assume that the other two coordinates are parallel to the remaining axes of the geometrical figure, in consequence of which the equation of the surface will become more simple, viz.

$$1 = \frac{x^2}{k^2} + \frac{y^2}{k'^2} + \frac{z^2}{k''^2},$$

the three semiaxes being  $k, k', k''$ , of which  $k$  is the axis of rotation.

Further, the figure of the fluid in equilibrium can be no other than a spheroid of revolution. Draw a plane through the axis of rotation and any point  $(x y z)$  in the surface of the fluid. This plane will contain that part of the attraction of the spheroid which is parallel to the axis of rotation, or to the coordinate  $x$ : it will also contain the centrifugal force directed at right angles from the axis of rotation. The same plane will also contain the resultant of the attractions parallel to  $y$  and  $z$ ; for if it did not, the resultant might be resolved into two forces, one contained in the plane, and the other perpendicular to it; and the force perpendicular to the plane would partly act in a direction touching the surface of the spheroid, which is inconsistent with the equilibrium of the fluid. Wherefore, the whole attractive force at any point in the surface of the spheroid is contained in a plane passing through the point and the axis of rotation; which obviously excludes ellipsoids with three unequal axes, and limits the figures of equilibrium to spheroids formed by the revolution of an ellipsis about the axis of rotation; and as the centrifugal force necessarily causes the equatorial diameter to be longer than the polar axis, it follows that the figure of the fluid in equilibrium can be no other than an oblate elliptical spheroid of revolution, of which the equation is

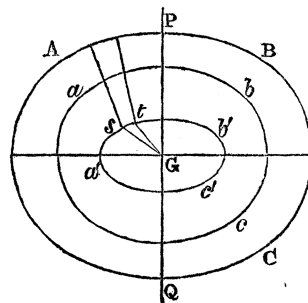
$$k^2 = x^2 + \frac{k'^2}{k^2} (y^2 + z^2),$$

the fluid turning about  $k$ , the less axis.

By the foregoing investigation, the problem for determining the equilibrium of a homogeneous planet in a fluid state is reduced to solving the equation of the upper surface, which is an expression of a known form, as the figure of the fluid is ascertained. The equation of the upper surface adjusts the oblateness of the spheroid to the quantity of the centrifugal force. It is only this part of the problem which, if we judge rightly, is fairly made out in the modes of investigation usually adopted; for in all these it is assumed that the figure of the fluid is an oblate elliptical spheroid, but, except in MACLAURIN'S demonstration, the equilibrium is not proved on satisfactory grounds. D'ALEMBERT first observed, that in general more spheroids than one may be in equilibrium with the same centrifugal force, or with the same velocity of rotation; and it is now well known that there may be two such spheroids, or one only, or that no spheroid of the proposed matter can be found that will be in equilibrium with the given quantity of centrifugal force. All this is pure mathematical deduction

from an algebraic equation; it is attended with no difficulty, and is very fully discussed by all the authors who have written on the figure of the earth; it would, therefore, be superfluous to treat of it here; but it may not be improper to add a few words for the purpose of explaining in what manner the number of solutions of the problem is limited by the nature of the equilibrium.

Let  $ABC$  represent an oblate elliptical spheroid of homogeneous fluid in equilibrium by revolving about the axis  $PQ$ ; and  $abc$ , an interior level surface, which is therefore similar to the upper surface  $ABC$ , and similarly posited about the centre: the stratum between the two surfaces will act upon the fluid within the level surface in two ways, namely, by pressure and by attraction. From the nature of the spheroid, the attraction of the stratum upon every particle within the level surface is zero; and the pressure of the exterior fluid acts upon every point of the same surface with equal intensity: wherefore, the whole mass  $ABC$  being in equilibrium, if the stratum be taken off, the remaining body of fluid  $abc$  will be in equilibrium separately. But another spheroid,  $a'b'c'$ , of a different form, may be traced within  $ABC$ , the less axes and the equators of the two figures coinciding, such that it will remain in equilibrium separately, upon abstracting the exterior fluid. Every small portion  $st$  of the surface  $a'b'c'$  is pressed inward by the exterior fluid; it also sustains a pressure from within outward, caused by the attraction of the fluid on the outside of the surface  $a'b'c'$  upon the particles within that surface. Now, although each of the two contrary pressures varies from one point of the surface to another, yet the spheroid may be so determined, that their joint action, or their difference, shall be the same at every point of the surface. When the spheroid  $a'b'c'$  has this figure, it will be in equilibrium with respect to the action of the exterior fluid; and, if that be abstracted, it will be in equilibrium separately, because the whole mass  $ABC$  is in equilibrium. What has been said may easily be proved by calculation; for the spheroid  $ABC$  being given, we know the pressure of the exterior fluid upon  $st$ ; we know also the attraction of the exterior fluid upon a particle of the spheroid  $a'b'c'$ , for it is equal to the difference of the attractions of the spheroids  $ABC$  and  $a'b'c'$  upon the particle: and hence it is easy to deduce, that the relation between the oblateness and the centrifugal force is expressed by the same equation in the spheroid  $a'b'c'$  and in the level surfaces.



It thus appears, that in general there are two spheroids of the same matter, but not more than two, which will be in equilibrium with the same rotatory velocity. If the oblateness of  $ABC$  increase, that of  $a'b'c'$  will decrease; and the two spheroids continually approaching the same figure, they will ultimately coincide in a limit at which there is only one form of equilibrium. On the other hand, as  $ABC$  becomes more

nearly spherical,  $a' b' c'$  will be more and more flattened ; so that, the centrifugal force being zero and  $A B C$  a perfect sphere,  $a' b' c'$  will be an infinitely thin circle of fluid particles in the plane of the equator.

The problem that has been solved leads to a consideration which it is important to notice, because it relates to a principle of equilibrium that has been very generally adopted. It has been shown that the equation of the surface (2.) is indeterminate, and admits of innumerable solutions ; but in every figure which satisfies that equation, the other equation (1.), viz.

$$p = \int \frac{dm}{f} + \frac{\varepsilon}{2} (y^2 + z^2) - C,$$

will hold at every interior point  $(x y z)$  of the mass of fluid. In this latter equation,  $p$  is the pressure of any canal issuing from the point  $(x y z)$  and extending to the surface of the fluid ; and therefore, in every figure which satisfies the equation of the surface, every such canal will exert the same pressure upon a molecule placed at the point  $(x y z)$ . Now of the innumerable figures that satisfy the equation of the surface there is only one that is in equilibrium ; and thus it is proved, that a mass of fluid, without being in equilibrium, may assume many figures in which every interior particle is pressed with equal intensity by all the canals issuing from it and terminating in the surface. And as neither the equation of the surface, nor the equal pressure of all the canals extending from a molecule to the surface, is sufficient to secure the equilibrium except when the forces are explicit functions of the coordinates ; so neither of the two properties can be employed in any other hypothesis respecting the forces, to verify an equilibrium, that is, to prove that a proposed figure will be in equilibrium.

8. In the following problem the forces in action are known functions of the coordinates, and the solution is deduced immediately from the theorem in No. 5.

## PROBLEM II.

To determine the figure of equilibrium of a fluid at liberty, the particles being supposed to attract one another with a force directly proportional to the distance, at the same time that they are urged by a centrifugal force caused by revolving about an axis.

As the attractions of the particles balance one another at the centre of gravity, in order to free that point from the action of any forces the axis of rotation must pass through it.

Let  $x, y, z$  denote the coordinates of an attracted particle, and  $x', y', z'$  those of an element  $dm$  of the mass, the origin being at the centre of gravity, and  $x, x'$  being parallel to the axis of rotation ; adopting for the unit of mass the whole given mass of fluid, and for the unit of force the attraction of the whole mass collected in a point upon a particle at the distance 1, the attraction of  $dm$  upon the assumed particle at

the distance  $f$  will be  $f dm$ ; and the cosines of the angles which  $f$  makes with  $x, y, z$  being

$$\frac{x - x'}{f}, \frac{y - y'}{f}, \frac{z - z'}{f},$$

the partial attractions, directed inward, and parallel to  $x, y, z$ , will be

$$dm (x - x'), dm (y - y'), dm (z - z');$$

and, by integrating, the sums of the like attractions of all the molecules of the mass are obtained, viz.

$$x \int dm - \int x' dm, y \int dm - \int y' dm, z \int dm - \int z' dm.$$

Now, by the property of the centre of gravity, we have

$$\int x' dm = 0, \int y' dm = 0, \int z' dm = 0;$$

wherefore, the attractions of the whole mass respectively parallel to  $x, y, z$  will be equal to

$$x \int dm, y \int dm, z \int dm,$$

or simply to  $x, y, z$ , because  $\int dm$  is the unit of mass.

Let  $\varepsilon$  denote the centrifugal force at the distance 1 from the axis of rotation, and estimated in parts of the unit of force; then the action of this force urging the particle in the prolongation of  $y$  and  $z$  will be equal to  $\varepsilon y$  and  $\varepsilon z$ .

Now, if  $X, Y, Z$  denote the whole accelerating forces acting parallel to  $x, y, z$ , we shall have

$$X = x, Y = (1 - \varepsilon) y, Z = (1 - \varepsilon) z;$$

which forces are therefore known functions of the point of action. Representing the intensity of pressure by  $p$ , we obtain

$$- dp = x dx + (1 - \varepsilon) \cdot (y dy + z dz);$$

and, by integrating,

$$C - p = \frac{x^2}{2} + (1 - \varepsilon) \cdot \frac{y^2 + z^2}{2},$$

which equation determines the pressure at the interior points of the fluid. The equation of the figure of the mass in equilibrium is obtained by making  $p = 0$ , viz.

$$C = \frac{x^2}{2} + (1 - \varepsilon) \cdot \frac{y^2 + z^2}{2}.$$

Supposing, therefore, that  $\varepsilon$  is less than 1, or that the centrifugal force at the distance 1 from the axis of rotation is less than the attraction of the mass collected in a point at the same distance, the fluid in equilibrium will have the form of an oblate elliptical spheroid of revolution.

As this problem is different from the first only in the law of attraction, it may be alleged that the methods of solution should be similar. There would be no difficulty in applying to it the same investigation employed in the first problem; but in whatever manner we proceed, the distinction between the two cases will remain unchanged.

In the second problem, the forces acting upon a particle within the surface are the same functions of the coordinates as the like forces acting upon a particle in the surface; because the forces which urge a particle in any situation depend only on the whole mass of fluid, and the distance of the particle from the centre of gravity. But in the first problem, if we except the particular class of figures susceptible of an equilibrium, the finding of which is an additional condition to be investigated, the forces urging a particle within the surface are not deducible from the forces at the surface merely by changing the coordinates of the point of action.

9. To complete the theory in this paper, it would be necessary to determine the figure of equilibrium of a revolving mass of homogeneous fluid, on the supposition that the particles attract one another with a force varying as any power of the distance. The solving of this problem would enable us to decide whether the equilibrium be possible in any other law of attraction but the direct proportion of the distance, or the inverse proportion of the square of the distance. The principles that have been laid down are sufficient to solve the problem enunciated in this general manner; but the application of them would require mathematical discussions too extensive to be entered upon at present. To conclude this paper, some observations will be made that seem to be called for by the notions that prevail on the subject of which it treats.

*On MACLAURIN'S Demonstration of the Equilibrium of the oblate elliptical Spheroid.*

In treating of the figure of the earth, NEWTON begins with observing that a homogeneous mass of fluid, supposing its particles urged only by their mutual attraction, would arrange itself in a form perfectly spherical. If this sphere acquire a revolving motion about one of its diameters, a new force will be impressed on its particles, causing them to recede from the axis of rotation; and, in obedience to this force, the fluid will subside at the poles and dilate itself in the direction parallel to the equator. NEWTON assumes, without alleging any reason in support of his assumption, that the revolving fluid will permanently settle in an oblate elliptical spheroid. Admitting tacitly that this is the figure of equilibrium, he proves that the relative dimensions of the spheroid depend upon the proportion of the centrifugal force to gravity at the equator; and this proportion being ascertained by experiment in the case of the earth, he finds that the equatorial diameter is to the polar axis as 230 to 229. The whole of this speculation, when published in the Principia, was entirely new; it involves many points of difficult investigation; and the ability has always been admired by which the difficulties are either overcome or evaded by ingenious approximations sufficiently exact and requiring the least possible calculation. But this splendid theory was incomplete till it should be proved that a fluid sphere turning upon an axis must assume the form of an elliptical spheroid. The attention of geometers was therefore turned to this point. The subject was treated by Mr. JAMES

STIRLING in 1735, and by CLAIRAUT in 1737, but only on the supposition of a spheroid little different from a sphere; and the results obtained by these geometers perfectly coincided with the determination of NEWTON. In a dissertation on the tides, which shared the prize of the Academy of Sciences of Paris in 1740, MACLAURIN made a great addition to the Newtonian theory, by proving that any proposed elliptical spheroid of homogeneous fluid would be in equilibrium if it revolved about its less axis with a certain rotatory velocity, and by introducing in his demonstration accurate notions respecting the conditions required for the equilibrium of a fluid entirely at liberty.

If an oblate elliptical spheroid of homogeneous fluid revolve about the less axis, the equilibrium of the mass will be secured if the resultant of the attractive and centrifugal forces acting upon a particle in the surface be directed perpendicularly towards the surface. In order to prove this, suppose that innumerable surfaces are described within the spheroid, similar to the upper surface, and similarly posited about the centre, and it will be easy to prove with respect to a particle in any of the interior surfaces, that the resultant of its centrifugal force, and of the attraction upon it of all the matter within the surface in which it is placed, is perpendicular to that surface. Now it is proved in the Principia that all the matter between the upper surface and any of the interior surfaces exerts no attraction upon a particle either in or within that surface; and hence it follows that the resultant of the centrifugal force of a particle, and the attraction upon it of all the matter of the spheroid, is perpendicular to the interior surface passing through the particle. The interior surfaces are therefore the true level surfaces of the spheroid, and the equilibrium of the revolving mass is established by the reasoning in the theorem in No. 5. From this demonstration it would appear that the Newtonian property, according to which the matter of a homogeneous stratum bounded by two similar and concentric elliptical surfaces does not attract a particle within the stratum, is not merely accidental to the equilibrium, but a condition necessary to its existence.

The equilibrium of the oblate spheroid may be made out by a different process. The attraction of the mass upon one of its particles may be investigated; and, when this done, it is found that the attractions parallel to the equator and perpendicular to the same plane, are proportional to the respective distances of the particle from the axis of rotation and from the equator. It thus appears that the forces urging any particle are known expressions of the coordinates of the point of action; and therefore the solution of the problem is immediately deduced from the theorem in No. 5. Now in this procedure there is no direct mention made of the Newtonian property; and hence it may, perhaps, be alleged that it is not essential to the equilibrium, although it is a principal step in the former demonstration. But a little reflection will show that the property in question is a condition no less necessary in this than in the former investigation; for it is by means of it that the forces acting upon a particle are disengaged from the upper surface of the fluid, the boundary

of the attracting mass, and are brought to depend entirely upon the situation of the particle with respect to the equator and the axis of rotation. This second investigation, therefore, concurs with the first, in proving that the Newtonian property is necessary to the equilibrium of the spheroid, and not merely accidental.

MACLAURIN's demonstration is different in some respects from either of the two investigations that have been mentioned. He requires three separate conditions for the equilibrium: first, the resultant of the centrifugal force and the attraction of the mass, must be perpendicular to the surface of the spheroid; secondly, every particle must be pressed equally in all directions; thirdly, all the columns reaching from the centre to the upper surface must balance and sustain one another. Now if the first of these conditions be fulfilled, and that too whether the mass of fluid be an elliptical spheroid or have any other figure, the other two will follow as necessary consequences. It may be observed further, that a demonstration proceeding on an arbitrary enumeration of properties, which may not be complete, makes a vague impression, and falls short of the conviction produced by a proof that rests on determinate principles bearing directly upon the point to be investigated. The conditions essential to MACLAURIN's demonstration are only these two: first, the attraction upon a particle proportional to its distances from the equator and the axis of rotation, which is peculiar to ellipsoids, and necessarily connected with the Newtonian property; secondly, the perpendicularity to the upper surface of the resultant of the forces acting upon a particle contained in that surface: and notwithstanding the beautiful train of reasoning employed by the author, his demonstration would gain in precision and clearness by omitting all that relates to the superfluous properties.

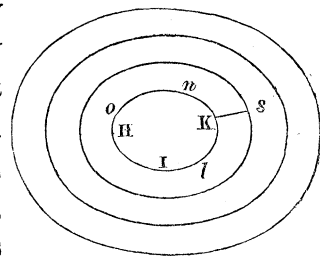
#### CLAIRAUT's *Theory*.

To CLAIRAUT belongs an important part of the theory of the figure of the earth. He was the first that entertained correct notions respecting the effect to alter the form of the terraqueous globe, produced by heterogeneity in its structure. At present we confine our attention to his general equations of the equilibrium of fluids, and their application to the case of a homogeneous planet. His theory is constructed with great analytical skill, and is seducing by its conciseness and neatness. From the single expression of the hydrostatic pressure are derived the equations of all the level surfaces, and of the upper surface of the fluid. But these equations are not sufficient in all cases to solve the problem. They are sufficient to solve it when the forces are known algebraic expressions of the coordinates of the point of action: they are not sufficient when the forces are not explicitly given, but depend, as in a homogeneous planet, on the assumed figure of the fluid. In this latter case, the solution of the problem requires, further, that the equations be brought to a determinate form by eliminating all that varies with the unknown figure of the fluid.

In the theory of CLAIRAUT it is tacitly assumed that the forces urging the interior particles are derived from the forces at the upper surface merely by changing the

coordinates of the point of action \*. Now there are cases, and the homogeneous planet is one, in which the forces acting on the interior particles are not deducible, in the manner supposed, from the forces at the surface; and with respect to such problems, the theory is silent, and has provided no means of solution.

But it will be satisfactory, and it is not difficult, to acquire just notions respecting CLAIRAUT's theory, by a careful examination of the principles as they are laid down by the author, for whose great abilities and high pretensions as a discoverer in science we entertain the sincerest respect, although we dissent from him on some points. The French geometer assumes for the foundation of his superstructure a mass of fluid,  $HKI$ , in equilibrium †. If  $f$  represent the force perpendicular to the surface of  $HKI$ , at any point  $K$ , and  $k$  the thickness  $Ks$  of an additional stratum  $onl$ ; and if the stratum be so determined that  $k \times f$  shall have constantly the same value at all the points of the surface; it will follow that the pressure of the stratum upon the surface on which it lies, is constant; and hence the body composed of the stratum and the original mass will be in equilibrium. In like manner, if a second stratum be added to the new body in equilibrium, the thickness being determined by the same condition as before, a third body of fluid in equilibrium will be obtained, consisting of two strata and the central mass. By adding more strata indefinitely, the dimensions of the mass of fluid may be enlarged to any extent, at the same time that the conditions of equilibrium are continually preserved. In all this it is evidently supposed that no change in the figure of the successive surfaces is effected by the strata laid upon them; for without this admission the procedure would be nugatory, and could lead to no determinate conclusion.



The investigation of CLAIRAUT is very elegant and geometrical, and carries with it the clearest evidence. It is entirely consonant to the theorem in No. 5. When it is not extended beyond its proper assumptions, it leads to a sure, and in truth to the only satisfactory principle of the equilibrium of a mass of fluid at liberty. It assumes that the pressure of every new stratum upon the surface on which it is laid, is caused solely by the forces in action at that surface, these forces being supposed to exert the same energy on all the particles of the infinitely small thickness of the stratum, and the thickness being so determined as to make the pressure constant. The procedure is agreeable to the usual rules of mathematical investigation, according to which the forces are conceived, not to flow continuously as the coordinates increase, but to vary from surface to surface by infinitely small gradations. Now this is very

\* When the forces acting upon the interior particles assume *singular forms* of expression at the surface, CLAIRAUT's theory fails; and this makes the distinction in the text necessary. But the whole theory, founded on an assumed principle, or upon an algebraic equation which determines the effect of the forces upon a particle taken individually, is so loosely delivered that it is difficult to speak of it with precision.

† Théorie de la Figure de la Terre, Première Partie, § xxi.

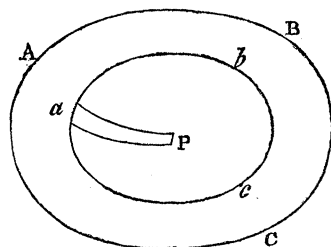
satisfactory when no cause of motion emanates from the fluid itself, and all the forces in action depend merely on the place of a particle. But if the fluid in question consist of attracting particles, will there not come into play the attraction of every additional stratum upon all the fluid contained within it, of which force no mention is made by CLAIRAUT? The only cause assigned for the pressure of the stratum upon the fluid below it, is the action of forces foreign to the matter of the stratum; the attraction of the stratum is inherent in that matter; the two causes of motion are distinct from one another, and their different effects ought to be separately considered. The procedure of CLAIRAUT, although it is unexceptionable when the forces in action depend only upon the position of a particle, seems chargeable with omission when applied to fluids consisting of particles that act upon one another by attraction or repulsion.

The initial body of fluid  $H K I$  is assumed to be in equilibrium; the equilibrium will not be disturbed by the pressure of the stratum  $o n l$ , which acts with equal intensity at every point of the surface  $H K I$ ; but if the fluid consist of attracting particles, the attraction of the stratum  $o n l$  upon all the particles contained within it may alter the form of the mass  $H K I$ , and the equality of pressure upon the changed figure no longer existing, the equilibrium will be destroyed. This argument has greater weight, because in the procedure of CLAIRAUT it is not the attraction of one stratum only which is neglected, but the sum of the attractions of all the successive strata, that is, no account is made of the attraction of a stratum of a finite thickness upon the particles within it.

It may perhaps be alleged that the attraction of a stratum upon the interior fluid is incomparably smaller than the forces which urge the particles of the stratum itself, and therefore that the first force may be accounted as nothing in respect of the other. Now the question is not about a comparison of forces different in degree, but whether the stratum attracting the particles within it in all directions, has power to move them and thereby to cause an alteration of figure. The procedure of CLAIRAUT, by making every stratum exert a constant pressure upon the fluid below it, leaves every particle of that fluid at liberty to obey the smallest impulse; and an equilibrium cannot subsist unless the attraction of the stratum be either absolutely zero, or cause a pressure urging every particle with equal intensity in all directions. If the stratum be bounded by concentric spherical surfaces, or by elliptical surfaces that are similar to one another and similarly posited, NEWTON has proved that the attraction of the stratum has no power to move the particles within it. Must these important propositions be extended, tacitly and without examination, to all strata, whatever be the bounding surfaces? If one bounding surface be spherical and the other elliptical, or if both be elliptical but dissimilar, will the attraction of the stratum be ineffective to move the interior particles? The plain truth is that CLAIRAUT has not attended to the attraction of the stratum, and consequently the application of his theory is limited to fluids consisting of particles that have no action upon one

another. The inadvertence with which the investigation of CLAIRAUT is chargeable, seems not to have been noticed, at least it is not remedied, by any of the authors who have subsequently handled the subject.

The attraction of the stratum being admitted, its effect becomes a subject for mathematical investigation. We may suppose a stratum of homogeneous fluid bounded by two surfaces of any figure,  $A B C$  and  $a b c$ ; and we may estimate the pressure tending to move an interior particle  $P$  in any direction, which is caused by the attraction of the stratum upon the contained fluid. Let  $dm$  represent an elementary portion of the stratum;  $x', y', z'$ , the coordinates of  $dm$ ;  $x, y, z$  those of  $P$ ; and  $f$  the distance between  $P$  and  $dm$ . The direct attraction of  $dm$  on  $P$  is equal to  $\frac{dm}{f^2}$ ; and the partial attractions tending inward parallel to  $x, y, z$ , are respectively



$$\frac{dm}{f^2} \cdot \frac{x - x'}{f}, \quad \frac{dm}{f^2} \cdot \frac{y - y'}{f}, \quad \frac{dm}{f^2} \cdot \frac{z - z'}{f},$$

or, which is the same,

$$\frac{dm}{f^2} \cdot \frac{df}{dx}, \quad \frac{dm}{f^2} \cdot \frac{df}{dy}, \quad \frac{dm}{f^2} \cdot \frac{df}{dz} :$$

and the total partial attractions on  $P$  of all the matter of the stratum will be

$$\frac{d \cdot \int \frac{dm}{f}}{dx}, \quad \frac{d \cdot \int \frac{dm}{f}}{dy}, \quad \frac{d \cdot \int \frac{dm}{f}}{dz},$$

the sign of integration extending to all the molecules of the stratum. Now if  $p$  represent the intensity of pressure, we shall have

$$dp = \frac{d \cdot \int \frac{dm}{f}}{dx} dx + \frac{d \cdot \int \frac{dm}{f}}{dy} dy + \frac{d \cdot \int \frac{dm}{f}}{dz} dz ;$$

and by integrating,

$$p = \int \frac{dm}{f} - C.$$

In this formula,  $\int \frac{dm}{f}$  is the sum of all the molecules of the stratum divided by their respective distances from  $P$ , and  $C$  is the like sum at any arbitrary point which may be assumed in the inner surface of the stratum at  $a$ , and may be joined to  $P$  by a narrow canal having any direction: and if we write  $\left[ \int \frac{dm}{f} \right]$  for  $C$ , that is, for the sum of all the molecules of the stratum divided by their respective distances from  $a$ , the value of  $p$  in the formula

$$p = \int \frac{dm}{f} - \left[ \int \frac{dm}{f} \right]$$

will be equal to the intensity of pressure urging the particle  $P$  in the direction of the canal. It appears, therefore, that the effect of the attraction of the stratum to move

the particle  $P$  is not infinitely little ; it is expressed by the difference of two definite integrals ; and, however small in degree the pressures urging  $P$  on different sides may be supposed, yet, if they be unequal, the particle must move in the direction in which the force is greatest. By omitting the attraction of the stratum, the procedure of CLAIRAUT is evidently defective, and applicable only to such fluids as consist of particles that have no action upon another.

But the investigation of CLAIRAUT, although limited as it is laid down by the author, when it is stated with all the generality of which it is susceptible, will be found on due reflection to contain the only true and satisfactory principle of the equilibrium of a mass of fluid at liberty\*. To render it perfectly general, nothing is wanting but to take into account all the forces necessary to complete the equilibrium at every separate stage of the procedure. The original mass  $H K I$  being supposed in equilibrium, the stratum  $onl$  must be adjusted as CLAIRAUT directs, so as to exert a constant pressure ; but a new condition must be added, that the body of fluid  $H K I$  be in equilibrium by the attraction of the stratum, that is, the pressures caused in the mass  $H K I$  by the attraction of the stratum, must urge every particle of it with the same intensity on all sides. When these conditions are fulfilled, the body of fluid, consisting of  $H K I$  and the stratum  $onl$ , will be in equilibrium, and its upper surface will be stable as was that of  $H K I$ , and capable of supporting additional strata. A new mass in equilibrium will be formed by adding a second stratum, so as to fulfill the same conditions as the first, that is, it must press with the same intensity at all points of the surface below it, and its attraction must have no power to move the particles contained within it. Continuing the same procedure and adding more strata indefinitely, a body of fluid of any dimensions will be formed, which is in equilibrium, all the forces in action being taken in account.

If we now examine a mass of fluid constructed by the foregoing process, so as to be in equilibrium, it is obvious that all the successive surfaces are deduced in the same manner from the forces acting on the particles contained in them. If the forces be explicit functions of the coordinates of their points of action, the condition that every surface must be pressed with the same intensity at all its points, determines the general equation of all the surfaces, nothing varying from one surface to another but the magnitude of pressure, as in the theorem in No. 5. The upper surface contains all the points of the fluid at which there is no pressure, and its equation alone ascertains the figure of equilibrium. This is the theory of CLAIRAUT in its full extent, and it is comprised in the theorem alluded to : but if the forces in action are not explicit functions of the coordinates, but depend upon the very figure to be investigated, the condition that the pressure must be constant in every successive surface, leads to an

\* It is obvious that all the steps of CLAIRAUT's procedure must be perfectly similar. As the central body  $H K I$  is supposed in equilibrium, so the addition of every stratum must produce a body in equilibrium, all the causes capable of moving a particle being taken into account ; if not, the process cannot be continued, or will fall into error.

equation that merely expresses a relation of two things alike unknown, namely, the figure of the fluid which is sought, and the forces resulting from that figure; and in this case it is necessary to take into account some other properties peculiar to the problem for the purpose of completing the solution. When the fluid consists of attracting particles, the equilibrium requires that the attraction of a stratum on the outside of any of the interior surfaces have no power to move the particles within that surface. Now it has been shown that the attraction of the stratum on the outside of the surface  $abc$ , causes a pressure,  $p$ , urging an interior particle at  $P$ , in the direction of a canal reaching from  $P$  to a point  $a$  in the surface  $abc$ , the quantity of which pressure is determined by the formula

$$p = \int \frac{dm}{f} - \left[ \int \frac{dm}{f} \right] :$$

and it is obvious that  $p$  will be the same to whatever point of the surface  $abc$  the canal is drawn, and consequently that the particle will have no tendency to move in any direction, if  $\left[ \int \frac{dm}{f} \right]$  have constantly the same value at all points of that surface. On the other hand, if  $\left[ \int \frac{dm}{f} \right]$  have different values at different points of the surface  $abc$ , the pressures upon  $P$  will be unequal, and the fluid will not be in equilibrium. Wherefore, in order to secure the equilibrium we must add to the constant pressure at all the points of every interior surface, as required by CLAIRAUT, or to the equation common to all these surfaces, this other condition, that the sum of the molecules of any stratum divided by their respective distances from a point in the inner surface of the stratum have constantly the same value at all the points of the surface. These conditions are the same with what has been investigated in the first part of this Paper; and, by means of the analysis in No. 7, they demonstrate that the figure of equilibrium of a homogeneous planet can be no other than an oblate elliptical spheroid of revolution.

In order fully to illustrate the investigation of CLAIRAUT, and to bring it completely within the power of the understanding, some further discussion is still required. The French geometer sets out with assuming, that the central mass  $HKI$  is in equilibrium; upon this all his inferences are grounded; but, in drawing the conclusion, he dismisses the first assumption, and substitutes for it the supposition that the central body of fluid is infinitely small. It may therefore be made a question, whether the results obtained are modified in any manner by the shifting of the original hypothesis.

The successive strata being so adjusted that the forces urging their particles are perpendicular to their surfaces, it is obvious that, upon every addition, the forces in action at the upper surface will be directed perpendicularly towards that surface, saving an abatement that must be made for the inequality of pressure upon the central mass, when that is not in equilibrium. But if the central mass be infinitely small, whether it be in equilibrium or not, will depend upon the action of very small

forces, and the effect of these to vary the direction of the forces in action at the successive upper surfaces from exact perpendicularity, will continually become less and less, and may be ultimately neglected. No objection can therefore be made to substituting, for the equilibrium of the central mass, the supposition that it is infinitely small, in so far at least as it is purposed to construct a body of fluid such that the forces in action at the upper surface shall be perpendicular to that surface.

If we suppose that the forces urging the particles of the fluid are expressed by known and explicit functions of the coordinates of their point of action, the body of fluid, as it acquires finite dimensions, will likewise approach continually to a known figure; for the equation of the surface, deduced from the perpendicularity of the forces, has a determinate form, which ascertains the figure of the mass when its volume is given. In this case, too, all the forces acting upon every individual stratum being taken into account, and the strata exerting a constant pressure upon one another, the equilibrium of a mass of fluid will be fulfilled simultaneously with the condition of the perpendicularity of the forces to the upper surface.

It remains to examine what will be the result when the central body  $H K I$ , supposed infinitely small and of any figure, consists of attracting particles. In this case there is no question about an equilibrium; because, although the forces at the successive upper surfaces are exactly estimated, CLAIRAUT has neglected the attraction of every stratum upon the body of fluid to which it is added, an omission which is fatal to an equilibrium of the mass. But as the procedure of that geometer always induces a figure which fulfills the condition of the perpendicularity of the forces to the upper surface, it is interesting to inquire whether, in the case of an attraction between the particles, the resulting figure is determinate and invariable, or indeterminate and varying with the figure of the small central body. Assume any body of finite dimensions similar to the small central mass  $H K I$ , and consisting of the same fluid; and supposing, for the sake of simplicity, that the law of attraction is that of nature, it is easy to prove, that the attractive forces acting in similar directions at similar points of the surfaces of the two bodies have constantly the same proportions as the linear dimensions of the bodies: and if the two bodies revolve with the same rotatory velocity about axes similarly placed, the centrifugal forces acting in similar directions at similar points of the surfaces, will likewise be to one another as the linear dimensions of the bodies. It appears, therefore, that the forces perpendicular to the surface of the central body  $H K I$ , although they are infinitely small, yet being proportional to the like forces at the surface of the finite body, they have given and finite proportions. Now upon the proportion of these forces depend the relative thickness and figure of the first additional strata at least; and as no limit can be assigned when this influence will cease, the conclusion undoubtedly is, that the ultimate surface will vary with the figure of the central mass. And thus the form induced by the procedure of CLAIRAUT upon a mass of fluid consisting of attracting particles is indeterminate, and susceptible of being varied indefinitely.

What has been said is well elucidated by the investigation that has been given of the exact figure of equilibrium, when all the forces in action are taken into account. Assuming that the problem is possible, it has been found that the supposition is verified, and all the conditions of equilibrium fulfilled, when that body is an oblate elliptical spheroid, and only when it has that figure. If the body  $H K I$ , whether its dimensions be finite or infinitely small, have the figure mentioned, and if the centrifugal and attractive forces be so adjusted that their resultant is, at every point, perpendicular to the surface of the spheroid, the procedure of CLAIRAUT will generate a series of figures all similar to one another, and all in equilibrium; but, as this proposition is exclusive, if we substitute for  $H K I$  a body of a different form, supposed infinitely small, none of the successive figures will be in equilibrium, although in the long run, when they have acquired finite dimensions, they will fulfill the condition of the perpendicularity of the forces to the upper surface.

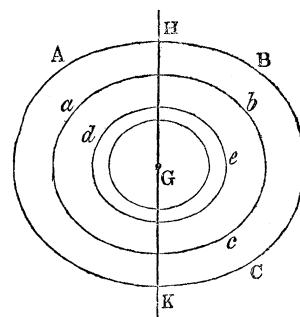
The discussion in which we have been engaged is of importance, because it shows the insufficiency of the methods usually employed for determining the equilibrium of a homogeneous fluid consisting of attracting particles. In this problem an equilibrium is not sufficiently established by making the upper surface perpendicular to the resultant of the forces acting upon the particles contained within it, nor by proving that all the narrow canals diverging from an interior particle, and terminating in the surface, press with equal intensity; nor can the problem be solved by attending solely to the forces that act upon the particles individually\*.

*On the Method of Investigation followed in the Paper published in the Philosophical Transactions for 1824.*

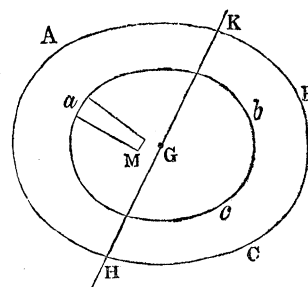
The equilibrium of a homogeneous planet may likewise be investigated by the method employed in my first paper on this subject, published in the *Philosophical Transactions* for 1824. As this method admits of being treated in few words, and will contribute to illustrate the principles on which the solution of the problem depends by placing them in a new light, I am induced to add a short explanation of it, more especially as it will give me an opportunity of stating clearly what is really liable to objection in that paper.

\* In a Memoir published in 1784, LEGENDRE has arrived at this conclusion, that the elliptical spheroid is exclusively the figure of equilibrium of a homogeneous planet. To the mathematical processes employed by that eminent geometer, no objections can be made. But, on examination, it will appear that the grounds on which his investigation really rests, are these two: first, the equation of the upper surface of the fluid, which is a necessary condition of equilibrium; secondly, an expression for the radius of the spheroid assumed arbitrarily and without reference to an equilibrium. Such a procedure can never be admitted as a complete and an *à priori* solution of the problem, unless it were first proved that every figure that can possibly fulfill the conditions of equilibrium is necessarily included in the expression assumed for the radius of the spheroid. No particular spheroid can be deduced from the equation of the upper surface alone, without first making a supposition respecting the expression of the radius: and this is an evident proof, that the equation is indeterminate and comprehends many different figures.

Let  $ABC$  represent a body of homogeneous fluid which revolves about the axis  $HK$ , passing through the centre of gravity  $G$ ; and describe an interior surface  $abc$ , similar to the upper surface  $ABC$ , and similarly posited about the point  $G$ : if we suppose that the mass  $ABC$  is in equilibrium by the action of the centrifugal force, and the attraction of its particles in the inverse proportion of the square of the distance, it is a property derived from the particular law of attraction and the nature of a centrifugal force, that every other body of the same fluid, as  $abc$ , similar to  $ABC$ , similarly posited about the common centre of gravity  $G$ , and revolving about the same axis  $KH$ , will be separately in equilibrium by the centrifugal force of its particles and the attraction of its own mass. It would be superfluous to repeat the demonstration of this proposition here, as it is attended with no difficulty, and has not been contested. And because the body of fluid  $abc$  is separately in equilibrium with respect to the centrifugal force of its particles and the attraction of its mass, it must likewise be in equilibrium with respect to the other forces that act upon it: for if it were not so, the whole body of fluid  $ABC$  would not be in equilibrium.



Now the only force external to the mass  $abc$ , and tending to change the figure of that mass, is the attraction of the exterior stratum upon the interior particles. Let  $M$  be any particle within the stratum: the several forces which act upon it are, first, the centrifugal force; secondly, the attraction of the mass  $abc$ ; and, thirdly, the attraction of the exterior stratum. On account of the separate equilibrium of the mass  $abc$ , the combined action of the two first forces has no tendency to move the molecule  $M$ ; and therefore the equilibrium of the whole mass  $ABC$  requires that the attraction of the exterior stratum be ineffective to move the same molecule. Thus every molecule  $M$  within the surface  $abc$  must be urged equally by the pressures which the attraction of the stratum causes in all canals originating at the molecule, and terminating in the surface  $abc$ . This is the same condition to which every other mode of investigation has led; and as the mathematical application of this property to determine the figure of equilibrium has already been fully detailed, it need not be repeated here.



In order to leave nothing unexplained, it will be proper to remark, that the interior surface  $abc$  is a level surface, that is, it is perpendicular at every point to the resultant of all the forces which act on a particle contained in it; for the centrifugal force of a particle at  $a$ , and the attraction upon it of the mass  $abc$ , have their resultant perpendicular to the surface  $abc$ , because the body of fluid  $abc$  is separately in equilibrium: and the attraction of the stratum upon the particle at  $a$  is perpendicular to the surface  $abc$ , because the sum of all the molecules of the stratum, divided

by their respective distances from any point in the surface  $abc$ , has the same invariable quantity. It follows from what is now proved, that the exterior fluid presses with the same intensity at every point of the interior surface  $abc$ .

The least attention to the internal pressures at the surface  $abc$ , and to the forces by which these pressures are caused, will show that the equilibrium of the mass  $ABC$  is secured by these two conditions: first, the resultant of the forces in action at the exterior surface must be directed perpendicularly towards that surface; and secondly, the level surfaces, that is, the interior surfaces, which are perpendicular to the resultant of all the forces acting upon the particles contained in them, must be similar to the outer surface, and similarly posited about the centre of gravity. These conditions of equilibrium, although enunciated in different terms, it will readily appear are not inconsistent with those before laid down, but are equivalent to them, and must necessarily bring out the same result.

The same things that have just been proved were investigated in the paper on this subject published in the Philosophical Transactions for 1824. There is no inaccuracy in that paper in deducing the conditions which the equilibrium requires to be fulfilled. These are, the perpendicularity to the upper surface of the resultant of the forces in action at that surface, and the immobility of a particle by the attraction of a stratum within which it is placed, and which is bounded by two surfaces similar and similarly posited to the upper surface. What is really exceptionable in that paper consists in the manner in which the second of the true conditions of equilibrium is conceived to be fulfilled. It is supposed in the paper that every individual particle within the stratum is attracted by the matter of the stratum so as to be drawn in all directions with equal intensity, which no doubt fulfills what is required, and is exact in particular figures; but being deficient in generality, it is an improper foundation on which to place the determination of the figure of equilibrium. To correct this misconception, it must be observed that the stratum, by attracting the particles within it, produces pressures in every part of the interior mass; and the immobility of a particle requires that it be pushed by the surrounding fluid with equal force in all directions. The difference between the two modes of action will be stated with most precision in mathematical language.

Assume a particle within the stratum,  $f$  being its distance from  $dm$ , a molecule of the stratum; the condition that the particle be attracted by the stratum equally in all directions, requires that the integral  $\int \frac{dm}{f^2}$ , extended to all the molecules of the stratum, have constantly the same value at all the points within the stratum; and the condition that the particle be at rest by the equal pressure of the surrounding fluid, requires that the same integral  $\int \frac{dm}{f^2}$  have a constant value at all the points of the lower surface of the stratum. The second determination, which admits the integral, although it must be constant in any one surface, to vary in any manner in

passing from one to another, is perfectly general; it embraces the full extent of the problem, and comprehends the first mode of action as a particular case. It happens that either of the two ways of rendering the attraction of a stratum ineffective to move the particles contained within it, leads precisely to the same final results in determining the figure of equilibrium of a homogeneous planet, which, although it does not excuse the misconception, makes the correcting of it less difficult. In conclusion, what is exceptionable in the paper of 1824 has already been explained publicly; and the paper in the Philosophical Transactions for 1831 is not liable to the same reproach.

APPENDIX, containing the Investigation of some Algebraic Formulas.

1. Development of  $\frac{1}{\sqrt{s^2 - 2sr\gamma + r^2}}$ , used in No. 7.

If we assume

$$s - rz = \sqrt{s^2 - 2sr\gamma + r^2},$$

the value of  $z$  will be

$$z = \gamma + \frac{1}{2} \cdot \frac{r}{s} (z^2 - 1):$$

now considering  $z$  as a function of  $\gamma$ , and applying the theorem of LAGRANGE, we deduce

$$z = \gamma + \frac{1}{2} \cdot \frac{r}{s} (\gamma^2 - 1) + \frac{1}{1 \cdot 2} \cdot \frac{1}{2^2} \cdot \frac{r^2}{s^2} \cdot \frac{d(\gamma^2 - 1)}{d\gamma} +, \&c.;$$

and by substituting this value in the assumed formula, we obtain

$$\sqrt{s^2 - 2sr\gamma + r^2} = s - r\gamma - \frac{1}{2} \cdot \frac{r}{s} (\gamma^2 - 1) - \frac{1}{1 \cdot 2} \cdot \frac{1}{2^2} \cdot \frac{r^2}{s^2} \cdot \frac{d(\gamma^2 - 1)}{d\gamma} -, \&c.;$$

then by differentiating with respect to  $\gamma$ , and dividing by  $sr$ , we finally obtain

$$\frac{1}{\sqrt{s^2 - 2sr\gamma + r^2}} = \frac{1}{s} + \frac{1}{2} \cdot \frac{r}{s^2} \cdot \frac{d(\gamma^2 - 1)}{d\gamma} + \frac{1}{1 \cdot 2} \cdot \frac{1}{2^2} \cdot \frac{r^2}{s^2} \cdot \frac{d d(\gamma^2 - 1)}{d\gamma} +, \&c.$$

This expression of the development is investigated differently in the Philosophical Transactions for 1824.

2. Demonstration of the theorem used in No. 7.

It is obvious that  $\theta$  and  $\theta'$  are the two sides of a spherical triangle,  $\varpi - \varpi'$  being the included angle,  $\psi$  the third side, and  $\sigma$  the angle opposite to  $\theta'$ ; wherefore, because  $\gamma = \cos \psi$  and  $\sqrt{1 - \gamma^2} = \sin \psi$ , we have by the known properties of spherical triangles,

$$\cos \theta' = \cos \theta \cdot \gamma + \sin \theta \cdot \sqrt{1 - \gamma^2} \cos \sigma$$

$$\sin \theta' \sin (\varpi - \varpi') = \sqrt{1 - \gamma^2} \sin \sigma$$

$$\sin \theta' \cos (\varpi - \varpi') = \sin \theta \cdot \gamma - \cos \theta \cdot \sqrt{1 - \gamma^2} \cos \sigma.$$

For the sake of brevity put  $\cos \theta = a$ ,  $\sin \theta \cos \varpi = \sqrt{1 - a^2} \cos \varpi = b$ ,  $\sin \theta \sin \varpi = \sqrt{1 - a^2} \sin \varpi = c$ ; and from the last expressions we readily deduce

$$a' = \cos \theta' = a \cdot \gamma + \sqrt{1 - a^2} \cdot \sqrt{1 - \gamma^2} \cos \sigma$$

$$b' = \sin \theta' \cos \varpi' = b \cdot \gamma - a \cos \varpi \cdot \sqrt{1 - \gamma^2} \cos \sigma + \sin \varpi \sqrt{1 - \gamma^2} \sin \sigma$$

$$c' = \sin \theta' \sin \varpi' = c \cdot \gamma - a \sin \varpi \cdot \sqrt{1 - \gamma^2} \cos \sigma - \cos \varpi \cdot \sqrt{1 - \gamma^2} \sin \sigma.$$

If these values be substituted in

$$a^m b^{m'} c^{m''},$$

and the several powers be expanded and reduced to terms containing the sines and cosines of the multiples of the arc  $\sigma$ , the result will be of this form :

$$\Gamma^{(0)} + (1 - \gamma^2)^{\frac{1}{2}} \cdot \Gamma^{(1)} \cos \sigma + (1 - \gamma^2)^{\frac{3}{2}} \cdot \Gamma^{(2)} \cos 2 \sigma +, \&c.$$

$$+ (1 - \gamma^2)^{\frac{1}{2}} \Delta^{(1)} \sin \sigma + (1 - \gamma^2)^{\frac{3}{2}} \Delta^{(2)} \sin 2 \sigma +, \&c.,$$

the expressions  $\Gamma^{(i)}$  and  $\Delta^{(i)}$  being integral functions of  $\gamma$ ; and it is to be observed that the index of the highest power of  $\gamma$  in  $\Gamma^{(0)}$  cannot exceed  $m + m' + m''$ . If we now multiply by  $d\sigma$  and integrate between the limits  $\sigma = 0$  and  $\sigma = 2\pi$ , we shall get

$$\int a^m b^{m'} c^{m''} \cdot d\sigma = 2\pi \times \Gamma^{(0)}.$$

Wherefore, attending to the expression of  $C^{(i)}$ , we have

$$\int -d\gamma d\sigma C^{(i)} a^m b^{m'} c^{m''} = \frac{1}{2^i} \cdot \frac{2\pi}{1 \cdot 2 \cdot 3 \dots i} \int -d\gamma \Gamma^{(0)} \frac{d^i(\gamma^2 - 1)^i}{d\gamma^i}.$$

Now it is easy to prove that

$$\int -d\gamma \cdot \gamma^n \cdot \frac{d^i(\gamma^2 - 1)^i}{d\gamma^i} = 0,$$

when  $n$  is less than  $i$ , the integral being taken between the limits  $\gamma = +1$  and  $\gamma = -1$ : and since the highest power of  $\gamma$  in  $\Gamma^{(0)}$  does not exceed  $m + m' + m''$ , it must be less than  $i$ ; and hence it follows that the integral under consideration is equal to zero.